Universidad Simón Bolívar.
Departamento de Matemáticas
Puras y Aplicadas.

26 de marzo de 2004.
Matemáticas II (MA–1112)

3^er Parcial. (35 %)
TIPO A

1. (3 puntos) Demuestre que $\operatorname{senh}(2x) = 2\operatorname{senh}(x)\cosh(x)$.

$$2\operatorname{senh}(x)\cosh(x) = 2\frac{e^x - e^{-x}}{2} \cdot \frac{e^x + e^{-x}}{2} = \frac{e^{2x} - e^{-2x}}{2} = \operatorname{senh}(2x)$$

---

2. (10 puntos) Calcule las siguientes integrales:

$$a)\ \frac{x^2+3x+3}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} = \frac{(A+B)x^2+(B+C)x+(A+C)}{x^3+x^2+x+1}$$

$$A+B=1, \quad B+C=3, \quad A+C=3 \qquad \Longrightarrow \qquad A=B=\frac{1}{2}, \quad C=\frac{5}{2}$$

$$\int \frac{x^2+3x+3}{(x+1)(x^2+1)}\,dx = \frac{1}{2}\int \frac{1}{x+1}\,dx + \frac{1}{2}\int \frac{x+5}{x^2+1}\,dx = \frac{1}{2}\int \frac{1}{x+1}\,dx + \frac{1}{2}\int \frac{x}{x^2+1} + \frac{1}{2}\int \frac{5}{x^2+1}\,dx.$$

$$= \frac{1}{2}\ln|x+1| + \frac{1}{4}\ln(x^2+1) + \frac{5}{2}\arctan(x) + C = \ln\left(\sqrt[4]{(x+1)^2(x^2+1)}\right) + \frac{5}{2}\arctan(x) + C.$$

b) Hacemos el cambio de variables:
$x = \sqrt{t}$,
$dx = \dfrac{dt}{2\sqrt{t}}$.

$$\int \frac{\tanh(\sqrt{t})\ln(\cosh(\sqrt{t}))}{\sqrt{t}}\,dt = 2\int \tanh(x)\ln(\cosh(x))\,dx = 2\int \frac{\operatorname{senh}(x)\ln(\cosh(x))}{\cosh(x)}\,dx =$$

Ahora, hacemos el cambio de variables:
$u = \cosh(x)$,
$du = \operatorname{senh}(x)dx$.

$$2\int \frac{\ln(u)}{u}\,du = \ln^2(u) + C = \ln^2(\cosh(\sqrt{t})) + C$$

---

3. Calcule los siguientes límites

$$a)\ \lim_{x\to 2} \frac{1}{x-2} - \frac{1}{\ln(x-1)} = \lim_{x\to 2} \frac{\ln(x-1)-x+2}{(x-2)\ln(x-1)} \overset{L'H}{=} \lim_{x\to 2} \frac{\frac{1}{x-1}-1}{(x-2)\frac{1}{x-1}+\ln(x-1)} = \lim_{x\to 2} \frac{1-(x-1)}{x-2+(x-1)\ln(x-1)} =$$

$$\lim_{x\to 2} \frac{2-x}{x-2+(x-1)\ln(x-1)} \overset{L'H}{=} \lim_{x\to 2} \frac{-1}{1+(x-1)\frac{1}{x-1}+\ln(x-1)} = \lim_{x\to 2} \frac{-1}{2+\ln(x-1)} = -\frac{1}{2}$$

$b)\ \lim_{x\to 0}\left(\frac{\tan(x)}{x}\right)^{\frac{1}{x^2}}$ hacemos $y = \left(\frac{\tan(x)}{x}\right)^{\frac{1}{x^2}} \Longrightarrow \ln(y) = \frac{1}{x^2}\ln(\frac{\tan(x)}{x})$.

$$\lim_{x\to 0} \frac{1}{x^2}\ln\left(\frac{\tan(x)}{x}\right) = \lim_{x\to 0} \frac{\ln\left(\frac{\tan(x)}{x}\right)}{x^2} \overset{L'H}{=} \lim_{x\to 0} \frac{\frac{x}{\tan(x)}\frac{x\sec^2(x)-\tan(x)}{x^2}}{2x} = \lim_{x\to 0} \frac{x}{\tan(x)} \lim_{x\to 0} \frac{x\sec^2(x)-\tan(x)}{2x^3} =$$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{\sec^2(x)+2x\sec^2(x)\tan(x)-\sec^2(x)}{6x^2} = \lim_{x\to 0} \frac{2x\sec^2(x)\tan(x)}{6x^2} = \lim_{x\to 0}\sec^2(x) \lim_{x\to 0} \frac{\tan(x)}{3x} = \frac{1}{3}$$

por lo tanto,

$$\lim_{x\to 0}\left(\frac{\tan(x)}{x}\right)^{\frac{1}{x^2}} = e^{1/3}.$$

4. (5 puntos) Halle el volumen del sólido generado al girar la región limitada por $y = 9-x^2$, $x \geq 0$, $y \geq 0$ alrededor de la recta $x = 3$.

$$\int_0^9 \pi(3^2 - \left(3 - \sqrt{9-y}\right)^2)\,dy = \pi \int_0^9 6\sqrt{9-y} - (9-y)\,dy$$

$$= \pi \left(-4(9-y)^{\frac{3}{2}} - 9y + \frac{y^2}{2}\right)\Big|_0^9 =$$

$$\pi\left(0 - 81 + \frac{81}{2} + 4(9)^{\frac{3}{2}} + 0 - 0\right) = \frac{135\pi}{2}.$$

$$\int_0^3 2\pi(3-x)(9-x^2)\,dx = 2\pi \int_0^3 (x^3 - 3x^2 - 9x + 27)\,dx =$$

$$2\pi\left(\frac{x^4}{4} - x^3 - 9\frac{x^2}{2} + 27x\right)\Big|_0^3 =$$

$$2\pi(\frac{81}{4} - 27 - \frac{81}{2} + 81) = 18\pi = \frac{135\pi}{2}.$$

---

5. (5 puntos) Diga si la integral $\int_0^{\frac{\pi}{2}} \text{cotan}(x)\,dx$ converge o diverge. En caso de que sea convergente, halle su valor.

$$\int_0^{\frac{\pi}{2}} \text{cotan}(x)\,dx = \lim_{a\to 0^+} \int_a^{\frac{\pi}{2}} \text{cotan}(x)\,dx = \lim_{a\to 0^+} \left(\ln(\text{sen}(x))\Big|_a^{\frac{\pi}{2}}\right) = \lim_{a\to 0^+} (\ln(\text{sen}(\pi/2)) - \ln(\text{sen}(a)))$$

$= -\lim_{a\to 0^+} \ln(\text{sen}(a)) = \infty$, luego la integral diverge.

---

6. (5 puntos) Halle el o los valores de C para que la integral $\int_0^\infty \frac{Cx}{1+x^2}\,dx$ sea convergente.

$$\int_0^\infty \frac{Cx}{1+x^2}\,dx = \lim_{b\to+\infty} C\int_0^b \frac{x}{1+x^2}\,dx = \lim_{b\to+\infty} C\left(\frac{1}{2}\ln(1+x^2)\Big|_0^b\right) = \lim_{b\to+\infty} C\left(\frac{1}{2}\ln(1+b^2)\right)$$

este límite existe solamente si $C = 0$.